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COMMENT

On the method of Chan and Lu for Abel's integral equation

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Abstract. For the method proposed recently by Chan and Lu for the inversion of Abel's integral equation, we show that (i) under a wide range of circumstances, it yields numerical schemes identical with those obtained from the direct discretisation of the standard inversion formulae; (ii) although it avoids the explicit numerical differentiation in the standard inversion formulae, it does not avoid the numerical difficulties inherent in the inversion; and (iii) it is still equivalent mathematically and numerically to a half-differentiation.

1. Introduction

In a recent paper, Chan and Lu (1981) examined the Abel integral equation (their equation (3) after the application of the transformation used to derive their equation (8))

$$\delta(p)=\frac{1}{2}\int_p^{\infty}\frac{U(q)}{(q-p)^{1/2}}\,\mathrm{d}q \tag{1}$$

and made the interesting observation that, after some standard manipulations, the known inversion formulae due to Abel (cf Lonseth 1977),

$$U(q)=-\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}q}\left(\int_q^{\infty}\frac{\delta(p)}{(p-q)^{1/2}}\,\mathrm{d}p\right) \tag{2a}$$

and

$$U(q)=-\frac{2}{\pi}\int_q^{\infty}\frac{\delta'(p)}{(p-q)^{1/2}}\,\mathrm{d}p, \qquad \delta'(p)=\frac{\mathrm{d}\delta(p)}{\mathrm{d}p} \tag{2b}$$

can be rewritten as

$$U(q)=-\frac{2}{\pi}\int_0^{\infty}\frac{\delta(q+u^2)-\delta(q)}{u^2}\,\mathrm{d}u. \tag{3}$$

They then used this result and an example to conclude that because it does not involve numerical differentiation, the solution (3) of Abel's integral equation is less susceptible to errors in the input data.

This approach for constructing numerical schemes for the inversion of Abel's equation appears to be new. It is interesting to note however that the Hilbert transform of

$$\Delta'(u) = 2u\delta'(u^2), \qquad \Delta'(u) = \mathrm{d}\Delta/\mathrm{d}u, \qquad \delta'(z) = \mathrm{d}\delta/\mathrm{d}z,$$

when evaluated at zero, yields

$$⨍_{-\infty}^{\infty} \frac{\Delta'(u)}{u}\,\mathrm{d}u = 2\int_0^{\infty} \frac{\delta(u^2)-\delta(0)}{u^2}\,\mathrm{d}u. \tag{4}$$

Such formulae are fundamental in tomography (cf Herman 1980, ch 8) and are usually evaluated by applying quadrature to the right-hand side of (4).

In this note, we show that, though the use of (3) can on occasions have advantages over the use of the standard inversion formulae, all are computationally equivalent in that (2*a*), (2*b*) and (3) each correspond to a half-differentiation of the data $\delta(p)$.

2. The numerical differentiation implicit in the method of Chan and Lu

Product integration is invariably used to generate finite difference schemes for the evaluation of (2*a*) and (2*b*). The idea is to choose an approximation $\bar{\delta}(p)$ for $\delta(p)$ of such a form that the resulting approximations

$$\bar{U}(q) = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}q}\left(\int_q^{\infty} \frac{\bar{\delta}(p)}{(p-q)^{1/2}}\,\mathrm{d}p\right) \tag{5a}$$

and

$$\bar{U}(q) = -\frac{2}{\pi}\int_q^{\infty} \frac{\bar{\delta}'(p)}{(p-q)^{1/2}}\,\mathrm{d}p \tag{5b}$$

can be evaluated analytically. Such approximations would include polynomials, piecewise polynomials and splines. If the manipulations of Chan and Lu are now applied to (5*a*), it follows that

$$\bar{U}(q) = -\frac{2}{\pi}\int_0^{\infty} \frac{\bar{\delta}(q+u^2)-\bar{\delta}(q)}{u^2}\,\mathrm{d}u. \tag{6}$$

This shows that many of the numerical schemes which can be constructed for the evaluation of (3), by replacing $\delta(\cdot)$ by suitable approximations $\bar{\delta}(\cdot)$, can be identical with those obtained from the direct discretisation of the standard inversion formulae; and hence, that (2*a*), (2*b*) and (3) yield equivalent numerical methods under a wide range of circumstances.

From a direct inspection of (3), it can be seen that the numerical differentiation explicit in (2*a*) and (2*b*) has not been removed but only transformed to a less explicit form. In fact, when u^2 is in the neighbourhood of the origin, the integrand of (3) defines a finite difference approximation to the first derivative of $\delta(p)$ at $p = q$ for all q. Thus, if the integrand is evaluated at the origin by extrapolation, then a finite difference approximation to $\delta'(p)$ at $p = q$ will result.

This fact is not that surprising when it is observed that (2*a*), (2*b*) and (3) are mathematically equivalent, and hence, that (3) must correspond to a half-differentiation since (2*a*) and (2*b*) do (cf Sneddon 1966).

Because mathematical equivalence does not imply computational equivalence, we examine the computational behaviour of (3) relative to that of (2b) and hence (2a). Using standard arguments (cf Anderssen 1976), it can be shown that, if finite difference methods with even grid spacing h are used to evaluate either (2a) or (2b), then errors in $\delta(p)$ are amplified locally by a factor of the order of $h^{-1/2}$. In addition, it is well known that, for finite difference approximations to the derivative of a function, the corresponding amplification factor is of the order of h^{-1}. This leads naturally to the conclusion that, from a numerical point of view, the inversion of the Abel integral equation (1) is as badly posed as a 'half-differentiation'.

We now confirm that (3) is computationally equivalent to the standard inversion formulae (2a) and (2b) by showing that the local amplification of errors in $\delta(p)$ is again $h^{-1/2}$ when (3) is evaluated using general quadrature formulae. In fact, if we take

$$\delta(p_k)=\delta_k, \qquad p_k=kh, \qquad k=0,1,\ldots, \qquad q=p_j,$$

then most quadrature approximations of (3) will take the form (cf Davis and Rabinowitz 1967)

$$\hat{U}_j=-\frac{2h^{-1/2}}{\pi}\left[\sum_{k=1}^{\infty} W_k\delta_{j+k}\left(\frac{(k+1)^{1/2}-k^{1/2}}{k}\right)-W_0\delta_j\right], \tag{7}$$

where the weights W_k (which can depend on j) are bounded and independent of h. Quadrature formulae based on (6) (or equivalently (5a) or (5b)) will also have this form if a piecewise polynomial approximation to δ is used.

3. Numerical performance when the data are observational

It should be clear from the discussion of § 2 that it is not the mathematical formulation of the inversion formulae but the numerical implementation that is important. However, for two independent numerical implementations, data can always be chosen so that one will perform better than the other. Thus, the numerical comparison of methods must be done with care to ensure that one is seeing the true character of their computational performance and not some data-dependent pathology.

To examine the question of numerical performance further, let us consider (7) when applied to observational data of the form

$$\tilde{\delta}_k=\delta_k+\varepsilon_k, \qquad k=0,1,2,\ldots,$$

where the $\delta_k=\delta(p_k)$ denote the exact data at the grid point p_k, and the ε_k are identically and independently distributed random variables with mean zero and variance σ^2.

If $\tilde{U}_j$ denotes the result of applying (7) to this data, it follows that

$$\tilde{U}_j=\hat{U}_j-\frac{2h^{-1/2}}{\pi}\left[\sum_{k=1}^{\infty} W_k^2\left(\frac{(k+1)^{1/2}-k^{1/2}}{k}\right)\varepsilon_{k+j}-W_0\varepsilon_j\right]$$

and hence

$$E(\tilde{U}_j)=\hat{U}_j$$

and

$$\operatorname{var}(\tilde{U}_j)=E[(\tilde{U}_j-\hat{U}_j)^2]=\frac{4\sigma^2}{\pi^2 h}\left[\sum_{k=1}^{\infty} W_k\left(\frac{(k+1)^{1/2}-k^{1/2}}{k}\right)^2+W_0^2\right].$$

To illustrate the wide range of values which $\mathrm{var}(\tilde{U}_j)$ can have, we consider the following three examples.

(i) Backward Euler quadrature with

$$W_0=\sum_{k=1}^{\infty}\left(\frac{(k+1)^{1/2}-k^{1/2}}{k}\right), \qquad W_k=1, \qquad k=1,2,\ldots,$$
$$\mathrm{var}(\tilde{U}_j)\doteqdot(1.62)4\sigma^2/\pi^2 h.$$

(ii) Trapezoidal quadrature, when the value

$$\lim_{u\to 0}\frac{\delta(p_j+u^2)-\delta(p_j)}{u^2}$$

is estimated by linear extrapolation. In this case,

$$W_0=\tfrac{1}{2}+\sum_{k=1}^{\infty}\left(\frac{(k+1)^{1/2}-(k-1)^{1/2}}{2k}\right),$$
$$W_1=1.5+\sqrt{2},$$
$$W_k=[(k+1)^{1/2}-(k-1)^{1/2}]/\{2[(k+1)^{1/2}-k^{1/2}]\}, \qquad k=2,3,\ldots,$$
$$\mathrm{var}(\tilde{U}_j)\doteqdot(5.60)4\sigma^2/\pi^2 h.$$

(iii) Product integration based on (4a), when $\delta(p)$ is approximated by a piecewise linear interpolating polynomial. Then

$$W_0=2,$$
$$W_k=2k[-(k+1)^{1/2}+2k^{1/2}-(k-1)^{1/2}]/[(k+1)^{1/2}-k^{1/2}],$$
$$\mathrm{var}(\tilde{U}_j)\doteqdot(5.43)4\sigma^2/\pi^2 h.$$

However, the choice of the quadrature scheme to use should not be based solely on the size of $\mathrm{var}(\tilde{U}_j)$. It is also necessary to take into account the behaviour of the discretisation error $\hat{U}_j-U_j$. Under suitable regularity assumptions about $\delta(p)$, this will decrease like $h^{1/2}$ for (i) and $h^{3/2}$ for (ii) and (iii). Unfortunately, the type of regularity conditions involved do not hold for certain applications such as axial tomography, since $U(q)$ can contain jump discontinuities. In such circumstances, it is probably best to choose schemes with small variance; but a detailed discussion is beyond the scope of this short note.

4. Conclusions

In conclusion, we make the following observation: all the above results are a consequence of the general result that computational difficulties associated with the inversion of improperly posed problems such as Abel's equation cannot be removed by simply manipulating the problem mathematically into an alternative form. Stabilisation can only be obtained through the introduction of additional structure (e.g. regularisation). For Abel's equation such methods have been discussed in some detail by Anderssen (1976), Anderssen and Jakeman (1975) and Wahba (1977).

Note 1. In some applications, $\delta(p)$ will be given at the points $p_k = (kh)^2$ rather than kh, $k = 0, 1, 2, \ldots$. To analyse such situations, it will be necessary to modify the quadrature formula (7). However, the basic conclusions made above will remain unchanged.

Note 2. In the above, we have glossed over the regularity conditions (e.g. $\delta(\infty) = 0$) which ensure that all the above steps are valid. Full details can be found in standard texts on analysis and integral equations. In addition, we have not worried about conditions which ensure that the summation in (7) exists, since, in applications, $\delta(p) \equiv 0$ for suitably large p.

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